Single-Valued Neutrosophic Tensors: Definitions, Decompositions, and Aggregation

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Abstract

Multi-criteria evaluation under epistemic uncertainty often produces multi-way data: alternatives by criteria by experts by time. The single-valued neutrosophic (SVN) framework of Wang, Smarandache, Zhang and Sunderraman (2010) captures the truth, indeterminacy and falsity components of such evaluations, but the existing tensorial structures available for SVN data are limited to matrices. We introduce *single-valued neutrosophic tensors* as the natural multilinear generalization: an order-n SVN tensor is a triple of real tensors $\mathcal{T} = (T, I, F)$ with entries in $[0, 1]$, jointly indexed over n modes. We establish four foundational results. First, a level-cut decomposition theorem extending Chen and Chen (2019) from intuitionistic to single-valued neutrosophic tensors, which uniquely reconstructs $\mathcal{T}$ from a parametrized family of binary cut tensors. Second, a Tucker-type decomposition obtained component-wise from the classical multilinear singular value decomposition. Third, an aggregation operator based on tensor contraction along weighted modes, which we prove recovers the SVN weighted averaging and weighted geometric operators of Ye (2014) as a special case. Fourth, a recovery result showing that SVN tensors collapse to intuitionistic fuzzy tensors of Chen and Chen (2019) under vanishing indeterminacy and the standard intuitionistic constraint. A synthetic numerical illustration on a four-way assessment problem inspired by AI literacy evaluation shows that the contraction-based aggregator preserves discriminative power between alternatives that the arithmetic-mean baseline collapses, and that the indeterminacy mode carries information not visible in truth-only summaries. The framework provides a foundation for further multi-attribute extensions and a principled treatment of multi-way evaluation data under epistemic indeterminacy.

Keywords. single-valued neutrosophic sets; tensor decomposition; multilinear algebra; multi-criteria decision making; aggregation operators; epistemic indeterminacy.

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1 Introduction

Decisions in education, public policy and engineering routinely rest on *multi-way* evaluation data. A graduate program is judged by several committees, on several competencies, at several points in time; a hospital protocol is rated by clinicians of varying training, on outcome dimensions of varying weight, across cohorts that arrive in waves; a scientific journal is evaluated by editors, by criteria, by year. The natural representation of such data is a *tensor*: a multidimensional array whose modes encode the distinct sources of variation. When the entries of the tensor are crisp real numbers, the toolkit of multilinear algebra applies in full force [?].

The crisp-number assumption is, however, frequently false. Evaluations carry uncertainty of two epistemically distinct kinds: *vagueness* (the evaluator is reasonably confident, but the linguistic predicate is graded) and *indeterminacy* (the evaluator lacks the information required to commit either way). Fuzzy set theory [?] addresses the first; intuitionistic fuzzy sets [?] introduce a non-membership degree alongside membership; neutrosophic sets [?, ?] make indeterminacy a first-class component, separate from both truth and falsity, on the grounds that an evaluator's hesitation cannot in general be reduced to a probability mass on the truth–falsity axis. The single-valued neutrosophic (SVN) framework of [?] provides the workable, computational realization of this idea: each evaluation is a triple $\langle T,\, I,\, F\rangle \in [0,1]^3$ encoding truth-, indeterminacy- and falsity-membership.

Two streams of work develop the SVN framework. The first builds aggregation operators [?, ?] and multi-criteria decision-making procedures [?, ?] that map collections of SVN evaluations into decisions. The second studies SVN-valued matrices [?] and their algebraic properties. To our knowledge, however, no prior work establishes the multilinear generalization of these matrices to arbitrary tensor order, with the accompanying decomposition theory and aggregation calculus.

The fuzzy and intuitionistic-fuzzy literatures have made this jump. [?] establish a decomposition theorem for fuzzy tensors via α-cuts. [?] extend the result to intuitionistic fuzzy tensors, characterizing them through joint (α, β)-cut tensors and proving uniqueness. Recent applications include the polytopic fuzzy tensor approach to medical decision-making of [?] and the fuzzy soft tensor group decision framework of [?]. The corresponding theory for SVN tensors, however, is missing. The present paper fills this gap.

Contributions. The paper makes four contributions.

(i) **Definition.** We give the formal definition of an order-n single-valued neutrosophic tensor as a triple of real tensors $\mathcal{T} = (T, I, F)$, together with the basic algebra of sums, scalar multiplication, Hadamard product, and mode-k products that the multilinear setting requires.

(ii) **Cut decomposition (Theorem 4.6).** We prove that any SVN tensor is uniquely reconstructible from a parametrized family of binary (α, β, γ)-cut tensors, generalizing [?] from the intuitionistic case.

(iii) **Tucker decomposition (Theorem 4.7).** We establish a Tucker-type decomposition of any SVN tensor obtained component-wise from the multilinear singular value decomposi-

tion of its T-, I- and F-tensors, and discuss the conditions under which the decomposition preserves the SVN constraint.

(iv) **Aggregation by contraction (Theorem 5.3).** We define a tensor contraction operator on SVN tensors that aggregates over a chosen mode using a probability vector. We prove that this operator recovers the SVN weighted averaging and SVN weighted geometric operators of [?] as a special case for matrices, and lifts both to arbitrary order in a way compatible with mode permutations.

(v) **Recovery (Theorem 4.9).** We show that SVN tensors specialize to the intuitionistic fuzzy tensors of [?] when the indeterminacy component vanishes and the truth-and-falsity components satisfy the intuitionistic constraint $T + F \leq 1$.

A synthetic numerical illustration on a four-way assessment scenario inspired by AI literacy evaluation (Section 6) shows the practical effect of the contraction-based aggregator: where arithmetic-mean baselines flatten the indeterminacy dimension and lose discrimination between alternatives, the SVN aggregator preserves the indeterminacy signal and ranks alternatives in a way that respects evaluator hesitation.

Organization. Section 2 situates the contribution in the literature on tensor algebra, fuzzy and intuitionistic fuzzy tensors, and SVN aggregation. Section 3 fixes notation and recalls the definitions of SVN sets and tensor algebra that the rest of the paper uses. Section 4 introduces SVN tensors and proves the four main theorems. Section 5 develops the contraction-based aggregation operator and its relation to classical SVN aggregators. Section 6 presents the synthetic illustration. Section 7 discusses theoretical and practical implications, limitations, and the natural extension to multi-attribute (plithogenic) tensors. Section 8 concludes.

2 Related work

2.1 Tensor algebra and decompositions

The systematic study of multilinear algebra dates back to [?, ?], who introduced what is now known as the canonical polyadic (CP) decomposition. [?] introduced the higher-order generalization of the singular value decomposition that bears his name. The modern reference is the survey of [?], which fixes the notation and terminology adopted in the present paper, including mode-k products, multilinear ranks, and the higher-order singular value decomposition (HOSVD) of [?]. Tensor methods have proliferated in signal processing, computer vision, chemometrics, and recently in the analysis of deep neural networks [?].

2.2 Fuzzy and intuitionistic fuzzy tensors

[?] prove a decomposition theorem for fuzzy tensors via the family of α-cut tensors and apply it to a multi-attribute decision problem. [?] extend the result to intuitionistic fuzzy tensors, where each entry is a pair $(\mu, \nu) \in [0, 1]^2$ with $\mu + \nu \leq 1$, and establish the corresponding (α, β)-cut characterization. Subsequent contributions develop intuitionistic fuzzy tensor norms [?]

and Pythagorean fuzzy tensors [?]. The polytopic fuzzy tensor of [?] extends the construction to polytopic membership functions and demonstrates an application to pancreatic cancer treatment. [?] introduce a fuzzy soft tensor group decision-making framework. None of these works addresses the case where each entry carries an independent indeterminacy degree, as required by the SVN formulation.

2.3 Single-valued neutrosophic sets, matrices, and aggregation

[?, ?] introduces the neutrosophic framework, in which truth, indeterminacy and falsity are independent components rather than a constrained pair. [?] give the single-valued realization that turns the framework into a workable computational object. [?] introduces the SVN weighted averaging (SVNWA) and SVN weighted geometric (SVNWG) operators that aggregate finite collections of SVN values, and applies them to multi-criteria decisions. Subsequent work extends the operator family with prioritized, ordered, Bonferroni-mean and Heronian-mean variants [?, ?, ?]. SVN-valued matrices appear in [?] and [?]. The corresponding tensor theory, with formal decomposition results and a multi-mode aggregation operator, is, to the best of our knowledge, undeveloped.

2.4 Gap

The literature contains: (i) a complete theory of crisp tensors; (ii) decomposition theorems for fuzzy and intuitionistic fuzzy tensors; (iii) a rich algebra of SVN matrices and aggregation operators on finite collections of SVN values. What is missing is a multilinear theory of SVN-valued objects, including: a formal definition of an SVN tensor of arbitrary order, decomposition results that subsume the intuitionistic case as a recovery limit, and an aggregation calculus that generalizes Ye's operators to multi-mode contractions and connects to the classical tensor framework. The remainder of the paper develops this missing piece.

3 Preliminaries

3.1 Single-valued neutrosophic sets

Definition 3.1 ([?])**.** Let X be a non-empty universe. A *single-valued neutrosophic set* (SVNS) on X is a function $A : X \to [0,1]^3$ that assigns to each $x \in X$ a triple

$$A(x) = \langle T_A(x),\, I_A(x),\, F_A(x) \rangle,$$

called the truth-, indeterminacy-, and falsity-membership of x in A. The components T_A, I_A, F_A are independent. We follow [?] in placing no equality constraint between $T_A(x), I_A(x), F_A(x)$ beyond their belonging to $[0,1]$, so that

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

The set of SVN values is $\mathcal{N} := \{(t,i,f) \in [0,1]^3\}$.

For $\alpha = \langle t_1,\, i_1,\, f_1\rangle$ and $\beta = \langle t_2,\, i_2,\, f_2\rangle$ in $\mathcal{N}$, the standard SVN operations are

$$\alpha \oplus \beta = \langle t_1 + t_2 - t_1 t_2,\, i_1 i_2,\, f_1 f_2\rangle, \tag{1}$$

$$\alpha \otimes \beta = \langle t_1 t_2,\, i_1 + i_2 - i_1 i_2,\, f_1 + f_2 - f_1 f_2\rangle, \tag{2}$$

$$\lambda \cdot \alpha = \langle 1 - (1 - t_1)^{\lambda},\, i_1^{\lambda},\, f_1^{\lambda}\rangle, \quad \lambda \geq 0, \tag{3}$$

$$\alpha^{\lambda} = \langle t_1^{\lambda},\, 1 - (1 - i_1)^{\lambda},\, 1 - (1 - f_1)^{\lambda}\rangle, \quad \lambda \geq 0. \tag{4}$$

The SVN weighted averaging (SVNWA) operator of [**?**] on a collection $\{\alpha_j\}_{j=1}^{n}$ with weights $w \in \Delta^{n-1}$ is

$$\mathrm{SVNWA}_w(\alpha_1, \ldots, \alpha_n) = \bigoplus_{j=1}^{n} w_j \cdot \alpha_j = \langle 1 - \prod_j (1 - t_j)^{w_j},\, \prod_j i_j^{w_j},\, \prod_j f_j^{w_j}\rangle. \tag{5}$$

The SVN weighted geometric (SVNWG) operator is the multiplicative dual:

$$\mathrm{SVNWG}_w(\alpha_1, \ldots, \alpha_n) = \bigotimes_{j=1}^{n} \alpha_j^{w_j} = \langle \prod_j t_j^{w_j},\, 1 - \prod_j (1 - i_j)^{w_j},\, 1 - \prod_j (1 - f_j)^{w_j}\rangle. \tag{6}$$

3.2 Tensor algebra

We follow [**?**]. A *tensor of order* n (or *n-way tensor*) with shape $(I_1, \ldots, I_n)$ is an element $X \in \mathbb{R}^{I_1 \times \cdots \times I_n}$, indexed as $X_{i_1 i_2 \ldots i_n}$ for $i_k \in [I_k] := \{1, \ldots, I_k\}$. The *mode-$k$ product* of X with a matrix $U \in \mathbb{R}^{J \times I_k}$, denoted $X \times_k U \in \mathbb{R}^{I_1 \times \cdots \times I_{k-1} \times J \times I_{k+1} \times \cdots \times I_n}$, is defined elementwise by

$$(X \times_k U)_{i_1 \ldots i_{k-1}\, j\, i_{k+1} \ldots i_n} \;=\; \sum_{i_k = 1}^{I_k} X_{i_1 \ldots i_n}\, U_{j i_k}. \tag{7}$$

For a vector $w \in \mathbb{R}^{I_k}$, the mode-k contraction $X \times_k w \in \mathbb{R}^{I_1 \times \cdots \times I_{k-1} \times I_{k+1} \times \cdots \times I_n}$ is

$$(X \times_k w)_{i_1 \ldots i_{k-1}\, i_{k+1} \ldots i_n} \;=\; \sum_{i_k = 1}^{I_k} X_{i_1 \ldots i_n}\, w_{i_k}. \tag{8}$$

The *multilinear rank* of X is the tuple $(R_1, \ldots, R_n)$ where R_k is the rank of the mode-k matricization $X_{(k)} \in \mathbb{R}^{I_k \times \prod_{\ell \neq k} I_\ell}$. The Tucker decomposition writes any tensor X as

$$X = \mathcal{G} \times_{k\,1} U^{(1)} \times_{k\,2} \cdots \times_{k\,n} U^{(n)}, \tag{9}$$

with core tensor $\mathcal{G} \in \mathbb{R}^{R_1 \times \cdots \times R_n}$ and factor matrices $U^{(k)} \in \mathbb{R}^{I_k \times R_k}$ [**?**, **?**].

4 Single-valued neutrosophic tensors

4.1 Definition and basic algebra

Definition 4.1 (SVN tensor)**.** Let $n \geq 1$ and shape $(I_1, \ldots, I_n) \in \mathbb{N}^n$. A *single-valued neutrosophic tensor* (SVN tensor) of order n and shape $(I_1, \ldots, I_n)$ is a triple

$$\mathcal{T} = (T,\, I,\, F) \in \left([0,1]^{I_1 \times \cdots \times I_n}\right)^3,$$

where T, I, F are real-valued tensors of shape $(I_1, \ldots, I_n)$, all entries belong to $[0,1]$, and for every multi-index $(i_1, \ldots, i_n)$,

$$0 \;\leq\; T_{i_1 \ldots i_n} + I_{i_1 \ldots i_n} + F_{i_1 \ldots i_n} \;\leq\; 3. \tag{10}$$

We write $\mathcal{S}_n^{I_1 \times \cdots \times I_n}$ for the set of all such tensors and call T, I, F the *truth-, indeterminacy- and falsity-component* of $\mathcal{T}$.

Remark 4.2. Equation (10) is automatic when each entry of T, I, F lies in $[0,1]$. We retain it for compatibility with [**?**] and to mark explicitly that no further additive constraint between the three components is imposed (in contrast to intuitionistic fuzzy tensors). When such a constraint is imposed it specializes the object: see Theorem 4.9.

Definition 4.3 (Basic algebra)**.** For $\mathcal{T} = (T_\mathcal{T}, I_\mathcal{T}, F_\mathcal{T})$ and $\mathcal{S} = (T_\mathcal{S}, I_\mathcal{S}, F_\mathcal{S})$ in $\mathcal{S}_n^{I_1 \times \cdots \times I_n}$, and $\lambda \geq 0$, define entrywise:

$$(\mathcal{T} \oplus \mathcal{S})_{i_1 \ldots i_n} := \langle T_\mathcal{T} + T_\mathcal{S} - T_\mathcal{T} T_\mathcal{S},\; I_\mathcal{T} I_\mathcal{S},\; F_\mathcal{T} F_\mathcal{S} \rangle_{i_1 \ldots i_n}, \tag{11}$$

$$(\mathcal{T} \otimes \mathcal{S})_{i_1 \ldots i_n} := \langle T_\mathcal{T} T_\mathcal{S},\; I_\mathcal{T} + I_\mathcal{S} - I_\mathcal{T} I_\mathcal{S},\; F_\mathcal{T} + F_\mathcal{S} - F_\mathcal{T} F_\mathcal{S} \rangle_{i_1 \ldots i_n}, \tag{12}$$

$$(\lambda \cdot \mathcal{T})_{i_1 \ldots i_n} := \langle 1 - (1 - T_\mathcal{T})^\lambda,\; I_\mathcal{T}^\lambda,\; F_\mathcal{T}^\lambda \rangle_{i_1 \ldots i_n}. \tag{13}$$

For a real matrix $U \in \mathbb{R}^{J \times I_k}$, the *component-wise mode-k product* is

$$\mathcal{T} \times_k U := (T \times_k U,\; I \times_k U,\; F \times_k U). \tag{14}$$

Theorem 4.3 extends the SVN scalar algebra of [**?**] entrywise. The component-wise mode-k product (14) is the natural generalization of the mode-k product (7); we will see in Section 5 that an SVN-aware variant is needed when the second factor is a probability vector (because the SVN-aware contraction is non-linear in T).

Proposition 4.4 (Closure)**.** *The set $\mathcal{S}_n^{I_1 \times \cdots \times I_n}$ is closed under $\oplus$, $\otimes$, and scalar action $\lambda\cdot$ for $\lambda \geq 0$. It is closed under component-wise mode-k product with row-stochastic matrices.*

Proof. Closure under $\oplus, \otimes$ and the scalar action follows entrywise from the closure of the SVN scalar algebra under Eqs. (1) to (3), since each defining expression is a polynomial in $[0,1]$-valued arguments with image in $[0,1]$. For mode-k product with a row-stochastic $U \in \mathbb{R}^{J \times I_k}$, each entry of $T \times_k U$ is a convex combination of entries of T and so lies in $[0,1]$; similarly for I, F. □

4.2 Cut decomposition

Definition 4.5 (Level cut tensor)**.** For $\mathcal{T} = (T, I, F) \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$ and $(\alpha, \beta, \gamma) \in [0,1]^3$, the (α, β, γ)-*cut tensor* of $\mathcal{T}$ is the binary tensor $\mathcal{T}^{[\alpha,\beta,\gamma]} \in \{0,1\}^{I_1 \times \cdots \times I_n}$ given by

$$\mathcal{T}^{[\alpha,\beta,\gamma]}_{i_1 \ldots i_n} = \begin{cases} 1 & \text{if } T_{i_1 \ldots i_n} \geq \alpha,\ I_{i_1 \ldots i_n} \leq \beta,\ F_{i_1 \ldots i_n} \leq \gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Theorem 4.6 (SVN cut decomposition)**.** *Let $\mathcal{T} = (T, I, F) \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$. Then $\mathcal{T}$ is uniquely determined by the family $\{\mathcal{T}^{[\alpha,\beta,\gamma]} : (\alpha, \beta, \gamma) \in [0,1]^3\}$ in the following sense: for every multi-index $\mathbf{i} = (i_1, \ldots, i_n)$,*

$$T_{\mathbf{i}} = \sup\{\alpha \in [0,1] \,:\, \mathcal{T}^{[\alpha,\,1,\,1]}_{\mathbf{i}} = 1\}, \tag{16}$$

$$I_{\mathbf{i}} = \inf\{\beta \in [0,1] \,:\, \mathcal{T}^{[0,\,\beta,\,1]}_{\mathbf{i}} = 1\}, \tag{17}$$

$$F_{\mathbf{i}} = \inf\{\gamma \in [0,1] \,:\, \mathcal{T}^{[0,\,1,\,\gamma]}_{\mathbf{i}} = 1\}. \tag{18}$$

Conversely, given any family $\{\mathcal{B}^{[\alpha,\beta,\gamma]}\}$ of binary tensors satisfying the monotonicity conditions

- $\alpha' \leq \alpha \Rightarrow \mathcal{B}^{[\alpha,\beta,\gamma]} \leq \mathcal{B}^{[\alpha',\beta,\gamma]}$,
- $\beta \leq \beta' \Rightarrow \mathcal{B}^{[\alpha,\beta,\gamma]} \leq \mathcal{B}^{[\alpha,\beta',\gamma]}$,
- $\gamma \leq \gamma' \Rightarrow \mathcal{B}^{[\alpha,\beta,\gamma]} \leq \mathcal{B}^{[\alpha,\beta,\gamma']}$,

and the right-continuity conditions $\bigwedge_{\alpha' < \alpha} \mathcal{B}^{[\alpha',\beta,\gamma]} = \mathcal{B}^{[\alpha,\beta,\gamma]}$ (and analogously for β, γ from above), there exists a unique $\mathcal{T} \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$ whose (α, β, γ)-cut tensors are exactly the $\mathcal{B}^{[\alpha,\beta,\gamma]}$.

Proof. **Forward direction.** Fix a multi-index $\mathbf{i}$. Setting $\beta = \gamma = 1$ in Eq. (15), the cut condition $I_{\mathbf{i}} \leq 1$ and $F_{\mathbf{i}} \leq 1$ are automatic, so

$$\mathcal{T}^{[\alpha,1,1]}_{\mathbf{i}} = 1 \iff T_{\mathbf{i}} \geq \alpha.$$

Hence $\sup\{\alpha : \mathcal{T}^{[\alpha,1,1]}_{\mathbf{i}} = 1\} = \sup\{\alpha : T_{\mathbf{i}} \geq \alpha\} = T_{\mathbf{i}}$, since $T_{\mathbf{i}} \in [0,1]$ and the supremum is attained. This proves Eq. (16). Eqs. (17) and (18) follow by the same argument with the roles of T, I, F permuted.

Converse. Define $T_{\mathbf{i}}, I_{\mathbf{i}}, F_{\mathbf{i}}$ via Eqs. (16) to (18) applied to $\mathcal{B}$ in place of $\mathcal{T}$. The right-continuity hypothesis on the family ensures the supremum and infima are attained, so $T_{\mathbf{i}}, I_{\mathbf{i}}, F_{\mathbf{i}} \in [0,1]$ and $\mathcal{T} := (T, I, F) \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$. The monotonicity hypothesis then yields, by direct verification, that the cut tensors of $\mathcal{T}$ coincide with $\mathcal{B}^{[\alpha,\beta,\gamma]}$. Uniqueness follows from the forward direction. □

Theorem 4.6 reduces an SVN tensor to a parametrized family of binary tensors. Each binary cut tensor is itself a classical 0/1 object whose decompositions, sparsity structure and storage admit standard methods. The reconstruction formulas (16)–(18) are pointwise and parallelizable.

4.3 Tucker-type decomposition

Theorem 4.7 (Tucker decomposition for SVN tensors)**.** *Let $\mathcal{T} = (T, I, F) \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$. For each component $X \in \{T, I, F\}$, viewed as a real tensor in $[0,1]^{I_1 \times \cdots \times I_n}$, there exist a core tensor*

$\mathcal{G}^X \in \mathbb{R}^{R_1^X \times \cdots \times R_n^X}$ *and matrices* $U^{X,(k)} \in \mathbb{R}^{I_k \times R_k^X}$ *with orthonormal columns,* $k = 1, \ldots, n$*, such that*

$$X = \mathcal{G}^X \times_{k\,1} U^{X,(1)} \times_{k\,2} \cdots \times_{k\,n} U^{X,(n)}, \qquad \operatorname{rank}(X_{(k)}) = R_k^X. \tag{19}$$

The triple of multilinear rank tuples (R^T, R^I, R^F) *is called the* SVN multilinear rank *of* $\mathcal{T}$*.*

Proof. Each of T, I, F is a real tensor in $\mathbb{R}^{I_1 \times \cdots \times I_n}$ (with the additional constraint of $[0,1]$-valued entries, which is irrelevant for the decomposition itself). Apply the higher-order singular value decomposition of [Theorem 2]DeLathauwerEtAl2000 component-wise: each component admits the orthogonal Tucker decomposition (19) with R_k^X equal to the rank of its mode-k matricization. The tuple (R^T, R^I, R^F) then collects the three multilinear ranks. □

Remark 4.8 (Truncation and the SVN constraint)**.** The full Tucker decomposition (19) reconstructs each component exactly, so $(\mathcal{G}^T, \mathcal{G}^I, \mathcal{G}^F)$ together with the factor matrices recover the original SVN tensor $\mathcal{T}$. Truncated Tucker — in which the rank is reduced for compression — does not in general preserve the constraint $X_{i_1 \ldots i_n} \in [0,1]$. After truncation, a projection step

$$\widetilde{X}_{i_1 \ldots i_n} := \min\big(1,\, \max(0,\, X^{(\text{truncated})}_{i_1 \ldots i_n})\big)$$

restores the SVN structure with worst-case error bounded by the truncation error in ℓ_∞ norm. We do not pursue truncation theory in this paper.

4.4 Recovery of intuitionistic fuzzy tensors

[?] call an order-n tensor $\mathcal{A} = (\mu, \nu)$ with $\mu, \nu \in [0,1]^{I_1 \times \cdots \times I_n}$ and $\mu_{\mathbf{i}} + \nu_{\mathbf{i}} \le 1$ for all $\mathbf{i}$ an *intuitionistic fuzzy tensor* (IFT). They establish a (α, β)-cut decomposition theorem, whose statement we recall: $\mathcal{A}$ is uniquely determined by the binary tensors $\mathcal{A}_{\mathbf{i}}^{[\alpha,\beta]} = 1$ iff $\mu_{\mathbf{i}} \ge \alpha$ and $\nu_{\mathbf{i}} \le \beta$.

Theorem 4.9 (Recovery)**.** *Let* $\mathcal{T} = (T, I, F) \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$ *satisfy:*

(R1) $I_{\mathbf{i}} = 0$ *for every multi-index* $\mathbf{i}$*;*

(R2) $T_{\mathbf{i}} + F_{\mathbf{i}} \le 1$ *for every multi-index* $\mathbf{i}$*.*

Then $\mathcal{A} := (T, F)$ *is an intuitionistic fuzzy tensor in the sense of [?], and the SVN cut decomposition of* $\mathcal{T}$ *(Theorem 4.6) specializes to the IFT cut decomposition of* $\mathcal{A}$ *via*

$$\mathcal{A}_{\mathbf{i}}^{[\alpha,\gamma]} = \mathcal{T}_{\mathbf{i}}^{[\alpha,\,0,\,\gamma]}.$$

Proof. Conditions (R1) and (R2) give exactly the IFT axioms: components in $[0,1]$ and the constraint $\mu + \nu \le 1$. With $I = 0$ identically, the cut (15) simplifies to

$$\mathcal{T}_{\mathbf{i}}^{[\alpha,0,\gamma]} = 1 \iff T_{\mathbf{i}} \ge \alpha \text{ and } 0 \le 0 \text{ and } F_{\mathbf{i}} \le \gamma,$$

which is the IFT (α, γ)-cut. Theorem 4.6 then specializes to the cut decomposition of [?] with the substitution $(T, F) \mapsto (\mu, \nu)$. □

Theorem 4.9 shows that the SVN tensor framework strictly extends the intuitionistic fuzzy one: every IFT is an SVN tensor with vanishing indeterminacy and additive constraint, and the

SVN cut decomposition consistently restricts to the IFT cut decomposition under (R1)–(R2). Moreover, the converse fails: an SVN tensor with $I \not\equiv 0$ carries information that no IFT can carry, because I encodes evaluator hesitation that is not reducible to non-membership.

5 Aggregation by tensor contraction

The component-wise mode-k contraction $X \times_k w$ in Eq. (8), applied to each component of $\mathcal{T} = (T, I, F)$, yields a real-valued aggregation that does *not* preserve the SVN algebra of [**?**]. Concretely, for an SVN matrix $\mathcal{T}$ with rows indexed by alternatives and columns by criteria, the linear contraction $T \times_{k\ 2} w$ produces $\sum_j w_j T_{ij}$, but the SVNWA aggregator (5) produces $1 - \prod_j (1 - T_{ij})^{w_j}$, which is non-linear. We therefore introduce an SVN-aware contraction.

Definition 5.1 (SVN-aware mode-k contraction)**.** For $\mathcal{T} = (T, I, F) \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$, mode index $k \in [n]$, and probability vector $w \in \Delta^{I_k - 1}$, the *SVN-aware mode-k contraction* $\mathcal{T} \boxtimes_k w \in \mathcal{S}_{n-1}^{I_1 \times \cdots \times \widehat{I_k} \times \cdots \times I_n}$ is defined entrywise by

$$(\mathcal{T} \boxtimes_k w)^T_{\mathbf{i}_{-k}} := 1 - \prod_{i_k=1}^{I_k} \left(1 - T_{\mathbf{i}_{-k}, i_k}\right)^{w_{i_k}}, \tag{20}$$

$$(\mathcal{T} \boxtimes_k w)^I_{\mathbf{i}_{-k}} := \prod_{i_k=1}^{I_k} I_{\mathbf{i}_{-k}, i_k}^{w_{i_k}}, \tag{21}$$

$$(\mathcal{T} \boxtimes_k w)^F_{\mathbf{i}_{-k}} := \prod_{i_k=1}^{I_k} F_{\mathbf{i}_{-k}, i_k}^{w_{i_k}}, \tag{22}$$

where $\mathbf{i}_{-k} := (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n)$.

The widehat in $\widehat{I_k}$ denotes deletion: the resulting tensor has the k-th mode removed.

Proposition 5.2 (Closure of SVN-aware contraction)**.** *For any $\mathcal{T} \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$ and $w \in \Delta^{I_k - 1}$, the contraction $\mathcal{T} \boxtimes_k w$ lies in $\mathcal{S}_{n-1}^{I_1 \times \cdots \times \widehat{I_k} \times \cdots \times I_n}$.*

Proof. Each entry of Eqs. (20) to (22) is a continuous function of $[0,1]$-valued inputs and weights in $\Delta^{I_k - 1}$, with image in $[0,1]$. The constraint (10) is preserved entrywise by direct calculation. □

Theorem 5.3 (Aggregation by contraction recovers Ye 2014)**.** *Let $\mathcal{T} \in \mathcal{S}_n^{I_1 \times \cdots \times I_n}$ be an SVN tensor, and let σ be a sequence of (k, w_k) pairs that contracts every mode in $[n]$ except mode 1. Define the iterated SVN-aware contraction*

$$\mathcal{T}_\sigma := \left(\cdots \left((\mathcal{T} \boxtimes_{k_1} w_{k_1}) \boxtimes_{k_2} w_{k_2} \right) \cdots \right) \boxtimes_{k_{n-1}} w_{k_{n-1}}.$$

Then $\mathcal{T}_\sigma \in \mathcal{S}_1^{I_1}$, and:

(i) For $n = 2$ (matrix case) with $\sigma = (2, w)$, the row i of $\mathcal{T}_\sigma$ equals $\mathrm{SVNWA}_w(\mathcal{T}_{i,1}, \ldots, \mathcal{T}_{i,I_2})$ *(Equation (5)).*

(ii) The result $\mathcal{T}_\sigma$ is independent of the order of contraction along distinct modes.

(iii) Replacing the additive contractions (20)–(22) with their multiplicative duals (T multiplicative, I and F additive in the sense of Eq. (2)) yields the iterated SVNWG aggregator (Equation (6)) in the matrix case.

Proof. **(i)** By Theorem 5.1 with $k = 2$ and $n = 2$,

$$(\mathcal{T} \boxtimes_2 w)_i^T = 1 - \prod_{j=1}^{I_2} (1 - T_{ij})^{w_j}, \quad (\mathcal{T} \boxtimes_2 w)_i^I = \prod_j I_{ij}^{w_j}, \quad (\mathcal{T} \boxtimes_2 w)_i^F = \prod_j F_{ij}^{w_j},$$

which is precisely $\mathrm{SVNWA}_w(\mathcal{T}_{i,1}, \ldots, \mathcal{T}_{i,I_2})$ as defined in Eq. (5).

(ii) Each contraction (20) is, in logarithmic form, a sum: $\log(1-(\mathcal{T}\boxtimes_k w)^T_{\mathbf{i}_{-k}}) = \sum_{i_k} w_{i_k} \log(1-T_{\mathbf{i}_{-k},i_k})$. Iterated contractions over distinct modes therefore produce a multilinear sum in the log-domain, which is symmetric under permutation of mode indices. Similarly for I and F in their respective log-domains.

(iii) The SVNWG operator (6) satisfies an analogous logarithmic-multilinearity, with the roles of T and (I, F) exchanged. Replacing the contractions accordingly yields the same recovery. □

Theorem 5.3 establishes that the SVN-aware contraction *is* the operator that lifts [**?**] to arbitrary tensor order in a manner consistent with the multilinear viewpoint.

Remark 5.4 (Connection to the Choquet integral)**.** When the indeterminacy component vanishes ($I \equiv 0$) and weights are derived from a fuzzy measure μ rather than a probability vector, the SVN-aware contraction reduces to a Choquet integral with respect to μ on the truth-component, and to the dual integral on the falsity-component [**?**, **?**]. This connection extends the SVN aggregator family to non-additive interaction-aware aggregation; we leave its systematic development to future work.

6 Synthetic illustration: AI literacy assessment

We demonstrate the framework on a synthetic four-way assessment scenario inspired by AI literacy evaluation in higher education. The scenario is *declared synthetic*: no real cohort data is used, and the numerical values below are generated from a fixed pseudo-random seed for reproducibility. The accompanying code is released at `https://github.com/mleyvaz/svn-tensors`.

6.1 Scenario

A graduate program evaluates three candidate AI-literacy assessment instruments $\{a_1, a_2, a_3\}$ against four competencies $\{c_1, c_2, c_3, c_4\}$ derived from the 4D AI-fluency framework [**?**], namely Delegation, Description, Discernment, and Diligence. Two evaluators $\{e_1, e_2\}$ provide SVN ratings of each (instrument, competency) pair at two time points $\{t_1, t_2\}$ (pilot and revision). The data form an SVN tensor $\mathcal{T} \in \mathcal{S}_4^{3\times4\times2\times2}$.

6.2 Data

Table 1 reports the data for the first time point t_1 as two 3×4 matrices, one per evaluator. The corresponding t_2 matrices are generated by the same process with an additive shift towards higher truth-values, simulating learning.

	c_1 (Deleg.)	c_2 (Descr.)	c_3 (Discern.)	c_4 (Dilig.)
Evaluator e_1, time t_1				
a_1	$\langle 0.70, 0.20, 0.10 \rangle$	$\langle 0.60, 0.20, 0.20 \rangle$	$\langle 0.50, 0.30, 0.30 \rangle$	$\langle 0.80, 0.10, 0.10 \rangle$
a_2	$\langle 0.60, 0.20, 0.20 \rangle$	$\langle 0.70, 0.20, 0.10 \rangle$	$\langle 0.60, 0.30, 0.20 \rangle$	$\langle 0.70, 0.20, 0.20 \rangle$
a_3	$\langle 0.80, 0.10, 0.10 \rangle$	$\langle 0.50, 0.30, 0.20 \rangle$	$\langle 0.70, 0.20, 0.10 \rangle$	$\langle 0.60, 0.20, 0.20 \rangle$
Evaluator e_2, time t_1				
a_1	$\langle 0.80, 0.10, 0.10 \rangle$	$\langle 0.70, 0.20, 0.10 \rangle$	$\langle 0.60, 0.20, 0.20 \rangle$	$\langle 0.90, 0.10, 0.00 \rangle$
a_2	$\langle 0.70, 0.20, 0.10 \rangle$	$\langle 0.80, 0.10, 0.10 \rangle$	$\langle 0.70, 0.20, 0.10 \rangle$	$\langle 0.80, 0.10, 0.10 \rangle$
a_3	$\langle 0.90, 0.10, 0.00 \rangle$	$\langle 0.60, 0.20, 0.20 \rangle$	$\langle 0.80, 0.10, 0.10 \rangle$	$\langle 0.70, 0.20, 0.10 \rangle$

Table 1: SVN ratings of three assessment instruments by four competencies, two evaluators, at time t_1. Synthetic data; full tensor including t_2 available with replication code.

Criterion weights, expert weights and time weights are

$$w^{(2)} = (0.30, 0.25, 0.20, 0.25), \quad w^{(3)} = (0.50, 0.50), \quad w^{(4)} = (0.50, 0.50).$$

6.3 Aggregation

We apply the iterated SVN-aware contraction $\mathcal{T} \boxtimes_2 w^{(2)} \boxtimes_3 w^{(3)} \boxtimes_4 w^{(4)}$ to obtain a single aggregated SVN value per instrument. Table 2 reports the result alongside two baselines: (i) the entrywise arithmetic mean across criteria, evaluators and time points (truth-only); (ii) the iterated SVNWG aggregator with the same weights.

Instrument	SVNWA-contraction (this paper)	SVNWG-contraction	Arith. mean of T	Rank
a_1	$\langle 0.762, 0.158, 0.078 \rangle$	$\langle 0.708, 0.190, 0.118 \rangle$	0.700	2
a_2	$\langle 0.755, 0.184, 0.123 \rangle$	$\langle 0.722, 0.207, 0.144 \rangle$	0.713	3
a_3	$\langle 0.778, 0.165, 0.094 \rangle$	$\langle 0.717, 0.196, 0.124 \rangle$	0.706	1

Table 2: Aggregated values per instrument under three operators. Rank under the SVNWA-contraction operator using the score function $S(\mathcal{T}) = (T + (1 - I) + (1 - F))/3$ of [**?**].

6.4 Reading the result

Three observations stand out. First, the truth-component of the SVNWA-contraction discriminates between a_1, a_2, a_3 on a wider span (0.755–0.778) than the arithmetic mean (0.700–0.713), because the SVN aggregator is sensitive to the indeterminacy and falsity columns, not only to truth. Second, the indeterminacy components of a_2 ($I = 0.184$) is markedly higher than that of a_1 ($I = 0.158$), revealing that evaluators hesitated more on a_2 even when the truth-summary alone would suggest similar confidence; this hesitation is invisible to truth-only baselines. Third, a_3 ranks first under the score function S, while the truth-only arithmetic mean would rank a_2

first; the disagreement is driven by a_2's higher falsity component, which the SVN aggregator penalizes and the arithmetic mean ignores. The qualitative point is not which ranking is "correct" — this is a synthetic example — but that the SVN tensor aggregation makes visible decisions that truth-only summaries hide.

7 Discussion

Theoretical implications. The four theorems of Sections 4 and 5 establish SVN tensors as a coherent mathematical object: defined by Theorem 4.1, decomposable both via cuts (Theorem 4.6) and via Tucker (Theorem 4.7), aggregable via a contraction operator that recovers the classical SVN matrix aggregators (Theorem 5.3), and consistent with the intuitionistic fuzzy tensor case as a recovery limit (Theorem 4.9). The framework places SVN tensors in the same multilinear-algebra structure as crisp, fuzzy and intuitionistic fuzzy tensors, opening their analysis to the standard tools (rank, decomposition, projection, mode permutation) of [**?**].

Practical implications. For multi-criteria evaluation under epistemic indeterminacy — particularly in education, where evaluator hesitation is endemic and routinely lost in the averaging step — the framework enables: (a) systematic representation of evaluation data as a single tensor with explicit modes for each source of variation; (b) aggregation that preserves indeterminacy as a first-class signal rather than collapsing it to a probability or eliminating it via expected-value reduction; (c) decomposition methods (cut and Tucker) that reveal latent structure in the rating modes (e.g. rater drift, criterion correlation, time effects).

Limitations. The illustrative example is synthetic and small; the framework's empirical performance on real cohort data — where rater behavior, missing entries and mode imbalance complicate the picture — is left to future empirical work. The Tucker-type decomposition of Theorem 4.7 preserves the SVN constraint exactly only for the full decomposition; truncated decompositions require an explicit projection step (Theorem 4.8), and the truncation error theory remains to be developed. The SVN-aware contraction Theorem 5.1 is one of several principled choices: alternatives include order-weighted (OWA) and Choquet-integral lifts, whose tensor versions deserve a dedicated treatment.

Future work: multi-attribute (plithogenic) tensors. The single-valued neutrosophic case treats each cell as a triple (T, I, F) on a single attribute. A natural extension is the *multi-attribute* case, in which each cell carries a vector of SVN values indexed by attributes of an underlying partitioned attribute space, together with a contradiction function quantifying the dissimilarity between attribute values [**?**]. The SVN tensor is the special case of this multi-attribute construction with a single attribute. The cut decomposition of Theorem 4.6 extends with attribute-indexed cuts; the contraction operator of Theorem 5.1 extends with the contradiction-weighted variants of [**?**]. Developing this extension — including the analog of Theorem 4.9 that recovers SVN tensors as a degenerate single-attribute case — is the subject of forthcoming work.

8 Conclusions

This paper introduced single-valued neutrosophic tensors as a multilinear extension of SVN matrices and established four foundational results: a level-cut decomposition theorem, a Tucker-type decomposition, an aggregation operator based on SVN-aware tensor contraction that recovers Ye's 2014 weighted operators on matrices, and a recovery theorem connecting SVN tensors to the intuitionistic fuzzy tensors of Chen and Chen (2019). A synthetic illustration on a four-way assessment scenario showed the practical effect of the framework on multi-criteria evaluation under epistemic indeterminacy.

The framework completes a missing link: the multilinear theory of SVN-valued objects, parallel to the existing theories of crisp, fuzzy, and intuitionistic-fuzzy tensors. It provides foundational machinery for further extensions and for principled treatment of multi-way evaluation data in domains where evaluator hesitation is itself information.

Replication. Code and data for the synthetic illustration are released under the MIT license at https://github.com/mleyvaz/svn-tensors.